50. Proposed by SYLVESTER ROBINS, North Branch Depot, New Jersey.

The edges of a rectangular parallelopiped are within 1 of the proportion $2:3:9$, and they are $2x\pm1$, $3x$ and $9x$, $(2x\mp1)^2+(3x)^2+(9x)^2$=the diagonal squared$=94x^2\mp4x+1=\square$. To find four integral values for x.

I. Solution by A. H. HOLMES, Box 963, Brunswick, Maine.

We may put it in the form : $90x^2+(2x\pm1)^2=\square$, or

$$m^2x^2-(m^2-90)x^2+(2x\pm1)^2=\square.$$

$\therefore\ 2m(2x\pm1)=(m^2-90)x$; $4mx\pm2m=m^2x-90x$.

$\therefore\ x=\pm(2m/(m^2-4m-90)$.

Let $m=nx$. Then $n^2x^2-4nx-90=\pm2n$; $n^2x^2-4nx+4=94\pm2n$.

Take plus sign and let $n=3$. $\therefore\ 3x=2+10=12$. $\therefore\ x=4$.

Now let $n=a/b^2$. $a^2x^2/b^4-4ax/b^2+4=94\pm2a/b^2=(94b^2\pm2a)/b^2$.

Now take $b=3$. $\therefore\ a=5/2$ and $a/b^2=5/18$.

$\therefore\ 5x/18=2+29/3$. $5x=36+174=210$. $\therefore\ x=42$.

Now let $b=10$. $\therefore\ a=9/2$ and $a/b^2=9/200$.

$\therefore\ 9x/200=2+97/10$. $9x=400+1940=2340$. $\therefore\ x=260$.

Now let $b=23$. $\therefore\ a=-3/2$ and $a/b^2=-3/1058$.

$\therefore\ -3x/1058=2-223/23$, or $3x=8142$. $\therefore\ x=2714$.

For $x=4$ we have : $94x^2+4x+1=\square$.

For $x=42$ we have : $94x^2-4x+1=\square$.

For $x=260$ we have : $94x^2+4x+1=\square$.

For $x=2714$ we have : $94x^2-4x+1=\square$.

II. Solution by A. H. BELL, Hillsboro, Illinois.

The equation readily reduces to : $t^2-94y^2=-90$..................(1), and $x=(t\mp2)/94$ ....(2). $(1)\div9$ gives $t'^2-94y'^2=-10$, and $t=3t'$, $y=3y'$..................(3).

One cycle.

$\sqrt{94}$. No. of Frac's:		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16:
Quotients complete Denom'rs	1 :	13	6	5	9	10	3	15	2	15	3	10	9	5	6	13	1:
Quotients	9 :	1,	2,	3,	1,	1,	5,	1,	(8),	1,	5,	1,	1	3	2	1	18:
Convergents $\frac{1}{0}$, $\frac{9}{1}$	:	$\frac{10}{1}$,	$\frac{29}{3}$,	$\frac{97}{10}$,	$\frac{126}{13}$,	$\frac{223}{23}$,	$\frac{1241}{128}$,	$\frac{1464}{151}$,	$\frac{12953}{1336}$,	$\frac{14417}{1487}$,	$\frac{85038}{8771}$,	$\frac{99455}{10258}$,	$\frac{184493}{19029}$,	$\frac{652934}{67345}$,	$\frac{1490361}{153719}$,	$\frac{2143295}{221064}$.—	

The convergents preceding the denominators, 10 of the complete quotients $=t'/y'=126/13$ and $85038/8771$ ....(4), as they are even fractions.

$\therefore$ answer the -10 of (3). To obtain other values of t' and y', take $v^2-94u^2=1$..................(5).

$$\left.\begin{array}{l}(3)\times(5)\text{ and }\pm188t'uvy'\quad (t'v\pm94uy')^2-94(t'u\pm vy')^2=-10\\ \qquad\qquad\text{or, } t_n'^2-94y_n'^2=-10\end{array}\right\}\text{.............(6).}$$

The smallest integral values for $v/u=2143295/221064$, but as fractional

values for t' and y' can be used as shown in (3), to obtain these we solve (5).

Let $v=v'/z$ and $u=u'/z$; then $v'^2-z^2=94u'^2$ ....(7).

Now let $u^2=pq$ and let 94=any two factors, then (7) can be made

$$\left.\begin{array}{l} v'+z=p^2 \text{ or } 2p^2 \\ v'-z=94q^2 \text{ or } 47q^2 \end{array}\right\}$$

add and subtract, etc. $v'=p^2+94q^2$ or $2p^2+47q^2$; $z=p^2-97q^2$ or $2p^2-47q^2$; $u'=2pq$..................(8).

In the right-hand values if $p=5$ and $q=1$, $v'=97$; $z=3$; $u'=10$. There are an infinite number of values but these are the only ones admissible.

(7) $v=97/3$ and $u=10/3$; substituting these along with those of (4) separately in (6) we have $t_n'=2/3$ and $24442/3$; and $t_n'=3946/3$ and $16493426/3$ with those in (4), will make six values for t', and now in (3) and (2) $x=0, 4, -42, 260, -2714$, and 175462, etc. The sign=side($2x\pm1$). $y=94, 39, 407, 2521, 26313$.

III. Solution by the PROPOSER.

This problem is suggested by a remark in No. 5, Vol. I.: "$x^2-94y^2=\pm1$; this is the most difficult number under 100."

1. Find initial terms in that infinite series of rational rectangular solids where the edges of each term are in proportion as 2 : 3 : 9, within 1 in the *thickness*.

Let $2x\pm1$, $3x$ and $9x$ be the edges ; then $94x^2\pm4x+1=\square=(mx\pm1)^2=m^2x^2\pm2mx+1$. $x=(\pm2m\mp4)/(94-m^2)$.

Say $m=\sqrt{(94)}=9/1$, $10/1$, $29/3$, $97/10$, $126/13$, $223/23$, $1241/128$, $1464/151$, etc.

When $m=$	10	29/3	97/10	223/2x	
Then $x=$	4	42	260	2714	
$2x\pm1=$	9	83	521	5427	Thickness.
$3x=$	12	126	780	8142	Width.
$9x=$	36	378	2340	24426	Length.
$\sqrt{(94x^2\pm4x+1)}=$	39	407	2521	26313	Solid diagonal.

2. Find first term in an infinite series of rational parallelopipeds where the dimensions of every solid are in proportion as 2 : 3 : 9, within 1 in the *width*.

Let $2x$, $3x\pm1$ and $9x$ represent the edges. Then $94x^2\pm6x+1=\square=(mx\pm1)^2=m^2x^2\pm2mx+1$. Whence $x=(2m\mp6)/(94-m^2)$, $m=\sqrt{(94)}=9, 10, 29/3, 97/10, 126/13$, etc.

$m=$	29/3	126/13
$x=$	24	429
$2x=$	48	858
$3x\pm1=$	73	1286
$9x=$	216	3861
Solid diagonal=	233	4159

3. Find a term in an infinite series of rational parallelopipeds where the edges are in proportion as 2 : 3 : 9, within unity in *length.*

Let $2x$, $3x$, and $9x\pm1$ be the edges. $94x^2\pm18x+1=\square=(mx\pm1)^2=m^2x^2\pm2mx+1$. $x=(2m\mp18)/(94-m^2)$. Substitute $m=1464/151$, and $x=15855$, $2x=31710$, $3x=47565$, $9x-1=142694$.

Proof: $31710^2+47565^2+142694^2=153719^2$.

4. Find some term in an infinite series of rational parallelopipeds where the dimensions come within 1 unit in the *thickness* of being in proportion as 3 : 6 : 7.

Let edges be $3x\pm1$, $6x$ and $7x$. $94x^2\pm6x+1=\square=(mx\pm1)^2=m^2x^2\pm2mx+1$. $x=(2m\mp6)/(94-m^2)$.

When $m=29/3$	$m=126/33$	
$x=$ 24	$x=$ 429	
$3x\pm1=$ 144	$3x\pm1=$ 1286	etc.
$6x=$ 144	$6x=$ 2574	
$7x=$ 168	$7x=$ 3003	
S. d. $=$ 233	S. d. $=$ 4159	

Proof: $73^2+144^2+168^2=233^2$.

5. Find some term in an infinite series of rational rectangular solids where the edges come within 1 unit in the *width* of being in the proportion of 3 : 6 : 7. Let the edges be represented by $3x$, $6x\pm1$ and $7x$. Then $94x^2\pm12x+1=\square=(mx\pm1)^2=m^2x^2+2mx+1$. $x=(2m\mp12)/(94-m^2)$. When $m=\sqrt{94}$ ......1464/151. Then $x=84258$ or 357870.

$3x=252774$	or $3x=1073610$
$6x-1=505547$	$6x+1=2147221$
$7x=589806$	$7x=2505090$
Diagonal $=816911$	Diagonal $=3469679$

6. Find a term in that infinite series of rational parallelopipeds wherein the edges of every solid are within unity in the *length* of being in proportion to each other as 3 : 6 : 7.

$(3x)^2+(6x)^2+(7x\pm1)^2=94x^2\pm14x+1=\square=(mx\pm1)^2$.

$94x\pm14=m^2x\pm2m$. $x=(2m\mp14)/(94-m^2)$. $m=\sqrt{94}$. Now when $m=29/3$, $x=60$, $3x=180$, $6x=360$, $7x-1=419$.

$180^2+360^2+419^2=581^2$.

Also solved by *J. H. DRUMMOND.*

51. Proposed by H. C. WILKES, Skull Run, West Virginia.

The difference between the roots of two successive triangular square numbers, [i. e. triangular numbers that are also square numbers], equals the sum of two successive integral numbers, the sum of whose squares will be a square number. Demonstrate. Or, if s and t be the roots of any two successive triangular number that are also square numbers, prove that $t-s=2n+1$, where $n^2(n+1)^2=\square$.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

$$\frac{n(n+1)}{2} \text{ is a square when } n=\frac{(1+\sqrt{2})^{2m}+(1-\sqrt{2})^{2m}-2}{4}.$$